A Space-Filling Polyhedron with Ring Geometry

A twelve-sided polyhedron which fills space with a single shape is described. Its shape can be created beginning with an octahedron of edge length 2 and boring a square 1x1 hole through it along one of its long axes. The name adopted here is "ring dodecahedron1".
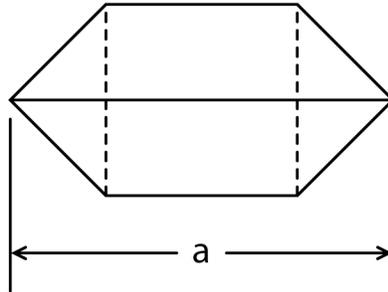

Figure 1. Side view of the ring dodecahedron. The lines trace its edges. The dashed lines show the hidden edges where the hole passes through it. The dimension a is the repeat period of the cubic unit cell.

The ring dodecahedron has twelve faces -- four are squares and eight are trapezoids. Each face is thus a four-sided polygon. It has twelve vertices, with four edges meeting at each vertex. All of the dihedral angles are either 45° or 90°. The unit cell volume is a3, and the volume of one ring dodecahedron is a3/6. There are six polyhedra in the cubic unit cell. Two of them lie in the xy plane, two in the yz plane, and two in the zx plane. All except the xy plane polyhedron located at 0,0,0 are shared with neighboring unit cells. The nonshaded rows of Table 1 show the centers x,y,z of the polyhedra in units of a, relative to the center of the unit cell.

Table 1. Locations and orientations of the polyhedra within the unit cell.

xy plane			yz plane			zx plane		
x	y	z	x	y	z	x	y	z
0	0	0	0	1/2	0	0	0	1/2
1/2	1/2	1/2	1/2	0	1/2	1/2	1/2	0

The ring dodecahedron is space-filling with a single shape. The basic motif can be illustrated with an xy-plane polyhedron. Its hole is threaded by two yz -plane polyhedra and two zx -plane polyhedra which precisely fill the hole. The hole in every polyhedron is threaded by four polyhedra lying in the other two planes. Thus each polyhedron is chain-linked to four other polyhedra. Even if this structure were constructed with a frictionless material it could not be pulled apart without exerting enough force to fracture the material.

Planes through the center of the cell perpendicular to the x, y, and z axes are all mirror planes. The cubic space group of this structure has not been determined.

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[1]We are not aware of any prior description of this space-filling polyhedron. Wolfram MathWorld has an informative page on space-filling polyhedra, but this one is not listed. It does abandon some of the usual rules in that it is not convex and it has ring topology.
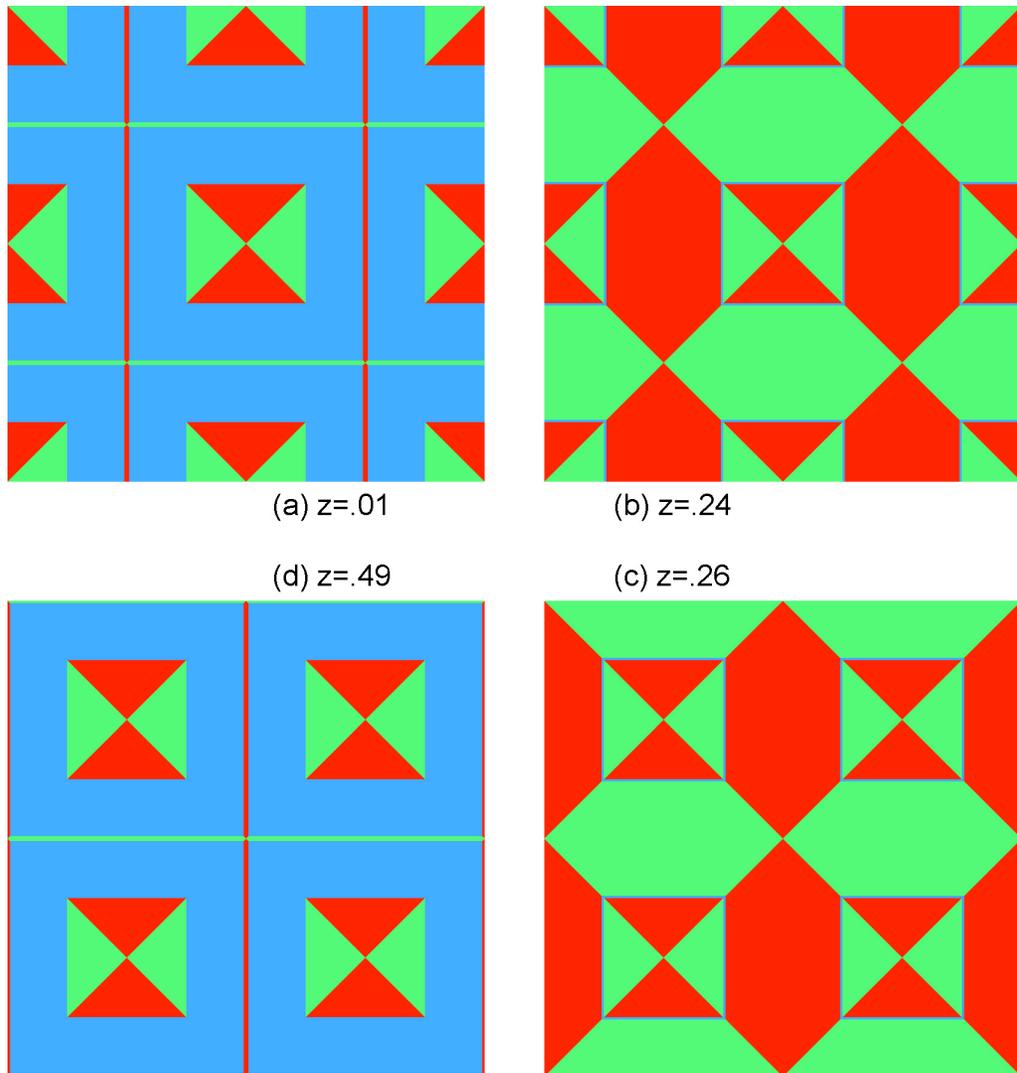
Figure 2. Cross sections in the xy plane at various z values, with z in units of the unit cell dimension a, relative to the center of the unit cell. A 2x2 array of cells is shown.

In Fig. 2(a) we see a plane just off the center of the cell. Two yz (red) and two zx (green) shapes thread through the blue xy polyhedron. Thin wedges of other yz and zx cells are seen between the blue xy shapes. Figure 2(b) shows a plane just less than halfway to the cell boundary. Only thin wedges of the blue xy shapes are seen. The large green and red images are near the holes in yz and zx shapes. In Fig. 2(c) we have moved up just a small amount to slightly more than halfway to the cell boundary. There is a sudden change from slicing through one set of yz and zx shapes to slicing through another set. Thin wedges of a different set of blue xy shapes are seen. Figure 2(d) shows the picture almost at the cell boundary. We see a different (offset) set of blue xy shapes centered half a cell width from those in Fig. 2(a), with yz and zx shapes threading through.